GCT535: Sound Technology for Multimedia

Fundamentals of DSP - Part 2

KAIST Graduate School of Culture Technology

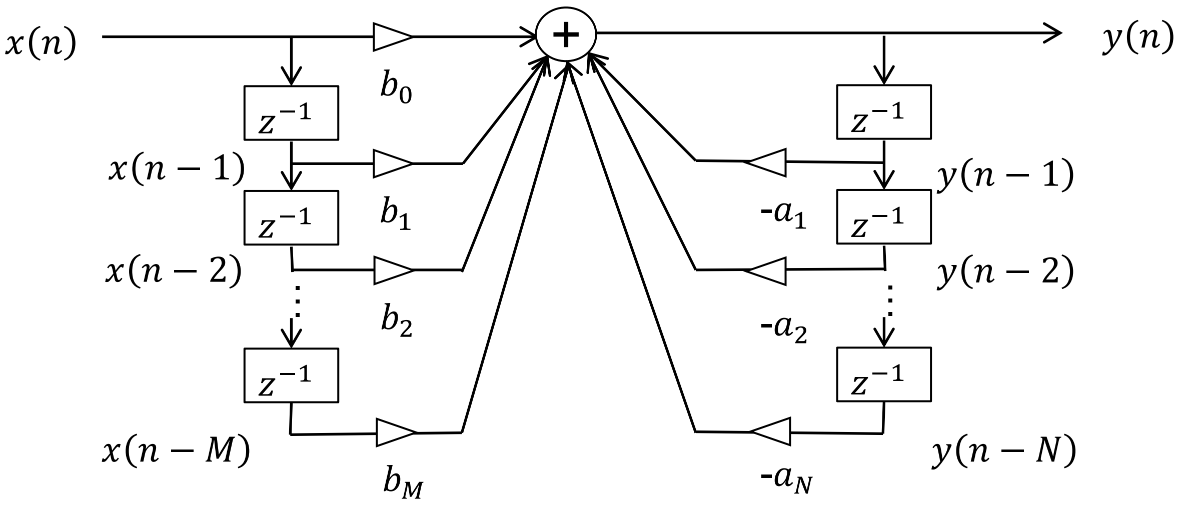
Juhan Nam

Goals

- Understanding frequency response using Z-transform and pole-zero analysis

General Form of LTI Systems

- Difference equation
  - $y(n) = b_0 \cdot x(n) + b_1 \cdot x(n-1) + b_2 \cdot x(n-2) + \ldots + \; b_M \cdot x(n-M)$
    $-a_1 \cdot y(n-1) - a_2 \cdot y(n-2) - \cdots - a_N \cdot y(n-N)$
- Signal flow graph



Frequency Response

- Sine-wave analysis
  - $x(n) = e^{j\omega n} \rightarrow x(n-m) = e^{j\omega(n-m)} = e^{-j\omega m}x(n)$ for any m
  - $y(n) = G(\omega)e^{j(\omega n+\theta(\omega))} \rightarrow y(n-m) = e^{-j\omega m}y(n)$ for any m
- Putting this into the different equation

$$y(n) = b_0 \cdot x(n) + b_1 \cdot e^{-j\omega} \cdot x(n) + b_2 \cdot e^{-j2\omega} \cdot x(n) + \ldots + b_M \cdot e^{-j\omega M} \cdot x(n) \\ -a_1 \cdot e^{-j\omega} \cdot y(n) - a_2 \cdot e^{-j2\omega} \cdot y(n) - \cdots - a_N \cdot e^{-j\omega N} \cdot y(n)$$

$$y(n) = \frac{b_0 + b_1 \cdot e^{-j\omega} + b_2 \cdot e^{-j2\omega} + \ldots + b_M \cdot e^{-j\omega M}}{1 + a_1 \cdot e^{-j\omega} + a_2 \cdot e^{-j2\omega} + \ldots + a_N \cdot e^{-j\omega N}} x(n)$$

$$H(\omega) = \frac{b_0 + b_1 \cdot e^{-j\omega} + b_2 \cdot e^{-j2\omega} + \ldots + b_M \cdot e^{-j\omega M}}{1 + a_1 \cdot e^{-j\omega} + a_2 \cdot e^{-j2\omega} + \ldots + a_N \cdot e^{-j\omega N}}$$

$H(\omega)$: frequency response

$G(\omega) = |H(\omega)|$: amplitude response

$\theta(\omega) = \angle H(\omega)$: phase response

Frequency Response

- In general, it is very complicated to obtain analytic expressions of the amplitude and phase responses from the frequency response
  - For example, $y(n) = x(n) + 0.9 \cdot y(n-1)$

    $H(\omega) = \frac{1}{1-0.9 \cdot e^{-j\omega}}$ → $G(\omega), \theta(\omega)$?

Z-Transform

- Z-transform
  - Define z as a variable in the complex plane: we call it z-plane
  - When $z = e^{j\omega}$ (on unit circle), the frequency response is a particular case of the following form

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2} + \ldots + \ b_M \cdot z^{-M}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} + \ldots + a_N \cdot z^{-N}}$$

  - We call this **z-transform** or **transfer function** of the filter
  - "z^{-1}" corresponds to one sample delay: delay operator or delay element
  - Digital systems are often expressed as z-transform: polynomial of z^{-1}
    - FIR: $H(z) = 1 + z^{-1}$, $H(z) = 1 - z^{-1}$
    - IIR: $H(z) = \frac{1}{1-0.9 \cdot z^{-1}}$, $H(z) = \frac{1}{1-0.9 \cdot z^{-1}+0.81 \cdot z^{-1}}$

Z-Transform

- More formally, z-transform is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

- With delay by M

$$X(z)z^{-M} = \sum_{n=-\infty}^{\infty} x(n-M) \cdot z^{-n} \qquad \text{(Shift Theorem)}$$

Z-Transform

- If this is applied to the difference equation of general form of LTI systems

$$Y(z) = b_0 \cdot X(z) + b_1 \cdot z^{-1}X(z) + b_2 \cdot z^{-2}X(z) + \ldots + \; b_M \cdot z^{-M}X(z)$$
$$-a_1 \cdot z^{-1}Y(z) - a_2 \cdot z^{-2}Y(z) - \cdots - a_N \cdot z^{-N}Y(z)$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2} + \ldots + \; b_M \cdot z^{-M}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} + \ldots + a_N \cdot z^{-N}} = H(z)$$

$Y(z) = H(z)X(z)$ (Convolution Theorem)

Z-Transform

- By replacing z with $e^{j\omega}$ back, z-transform is represented with

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n}$$

- Note that the the frequency response is discrete-time Fourier transform of impulse response
  - Also, note that the convolution theorem works for discrete-time Fourier transform or discrete Fourier transform

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \qquad Y(k) = H(k)X(k)$$

Poles and Zeros in Z-Transform

- The polynomial of z^{-1} in $H(z)$ can be factorized
  - We can find roots for both numerator and denominator
  - Zeros: roots of numerator
  - Poles: roots of denominator

$$H(z) = \frac{B(z)}{A(z)} = \frac{(1-q_1z^{-1})(1-q_2z^{-1})(1-q_3z^{-1})\ldots(1-q_Mz^{-1})}{(1-p_1z^{-1})(1-p_2z^{-1})(1-p_3z^{-1})\ldots(1-p_Nz^{-1})}$$

- We can analyze frequency response more easily using poles and zeros than numerator or denominator coefficient

Pole-Zero Analysis: Amplitude Response

- Amplitude Response
  - Computed using distances between poles and unit circles and distances between zeros and units circles on Z-plane

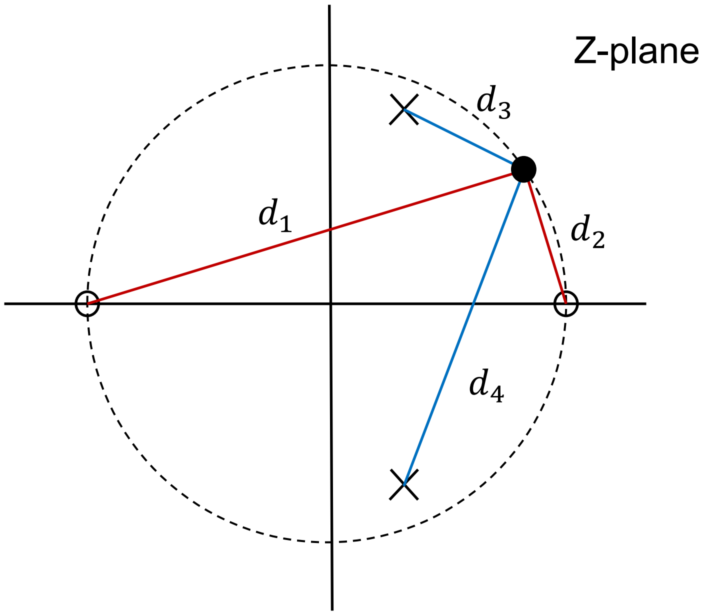
$$G(\omega) = |H(\omega)| = \left|\frac{B(\omega)}{A(\omega)}\right| = \left|\frac{\left(1 - q_1 e^{-j\omega}\right)\left(1 - q_2 e^{-j\omega}\right)\left(1 - q_3 e^{-j\omega}\right)\ldots\left(1 - q_M e^{-j\omega}\right)}{\left(1 - p_1 e^{-j\omega}\right)\left(1 - p_2 e^{-j\omega}\right)\left(1 - p_3 e^{-j\omega}\right)\ldots\left(1 - p_M e^{-j\omega}\right)}\right|$$

$$|H(\omega)| = \left|\frac{B(\omega)}{A(\omega)}\right| = \left|\frac{\left(e^{j\omega} - q_1\right)\left(e^{j\omega} - q_2\right)\left(e^{j\omega} - q_3\right)\ldots\left(e^{j\omega} - q_M\right)}{\left(e^{j\omega} - p_1\right)\left(e^{j\omega} - p_2\right)\left(e^{j\omega} - p_3\right)\ldots\left(e^{j\omega} - p_M\right)}\right|$$

$$|H(\omega)| = \left|\frac{B(\omega)}{A(\omega)}\right| = \frac{\left|\left(e^{j\omega} - q_1\right)\right|\left|\left(e^{j\omega} - q_2\right)\right|\left|\left(e^{j\omega} - q_3\right)\right|\ldots\left|\left(e^{j\omega} - q_M\right)\right|}{\left|\left(e^{j\omega} - p_1\right)\right|\left|\left(e^{j\omega} - p_2\right)\right|\left|\left(e^{j\omega} - p_3\right)\right|\ldots\left|\left(e^{j\omega} - p_M\right)\right|}$$

Example: Reson Filter

- Difference equation
  - $y(n) = x(n) + 2r \cdot \cos\theta \cdot y(n-1) - r^2 \cdot y(n-2) - x(n-2)$
- Transfer function
  - $H(z) = \frac{1-z^{-2}}{1-2r\cos\theta \cdot z^{-1} + r^2 \cdot z^{-2}}$
  - Poles: $r(\cos\theta + j\sin\theta), r(\cos\theta - j\sin\theta)$
  - Zeros: 1, -1
- Amplitude response
  - $G(\omega) = |H(\omega)| = \left|\frac{B(\omega)}{A(\omega)}\right| = \frac{d_1(\omega)d_2(\omega)}{d_3(\omega)d_4(\omega)}$



Poles and Stability

- If Poles are inside the unit circle
  - The system is stable
  - $r < 1$: the impulse response decays with oscillation
- If Poles are on the unit circle
  - The system is critically-stable
  - $r = 1$: the impulse response oscillates with constant amplitude (sine generation)
- If Poles are outside the unit circle
  - The system is unstable (e.g. howling when mics are closely placed near speakers)
  - $r > 1$: the impulse response diverges

Pole-Zero Analysis: Phase Response

- Phase Response
  - Computed using angles between poles and unit circles and angles between zeros and units circles on Z-plane

$$\theta(\omega) = \angle H(\omega) = \frac{\angle B(\omega)}{\angle A(\omega)} = \frac{\angle\left(1 - q_1 e^{-j\omega}\right)\left(1 - q_2 e^{-j\omega}\right)\left(1 - q_3 e^{-j\omega}\right) \ldots \left(1 - q_M e^{-j\omega}\right)}{\angle\left(1 - p_1 e^{-j\omega}\right)\left(1 - p_2 e^{-j\omega}\right)\left(1 - p_3 e^{-j\omega}\right) \ldots \left(1 - p_M e^{-j\omega}\right)}$$

$$= \angle\left(1 - q_1 e^{-j\omega}\right) + \angle\left(1 - q_2 e^{-j\omega}\right) + \angle\left(1 - q_3 e^{-j\omega}\right) + \cdots + \angle\left(1 - q_M e^{-j\omega}\right)$$

$$-\angle\left(1 - p_1 e^{-j\omega}\right) - \angle\left(1 - p_2 e^{-j\omega}\right) - \angle\left(1 - p_3 e^{-j\omega}\right) - \cdots - \angle\left(1 - p_M e^{-j\omega}\right)$$

$$\theta(\omega) = \angle H(\omega) = \frac{\angle B(\omega)}{\angle A(\omega)} = \angle\left(e^{j\omega} - q_1\right) + \angle\left(e^{j\omega} - q_2\right) + \angle\left(e^{j\omega} - q_3\right) + \cdots + \angle\left(e^{j\omega} - q_M\right)$$

$$-\angle\left(e^{j\omega} - p_1\right) - \angle\left(e^{j\omega} - p_2\right) - \angle\left(e^{j\omega} - p_3\right) - \cdots - \angle\left(e^{j\omega} - p_M\right)$$

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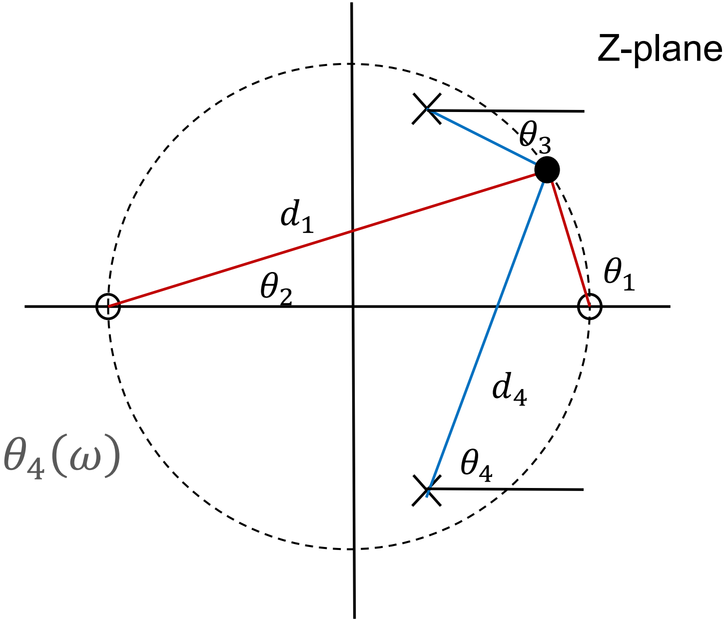
- Transfer function
  - $H(z) = \frac{1-z^{-2}}{1-2r\cos\theta \cdot z^{-1} + r^2 \cdot z^{-2}}$
  - Poles: $r(\cos\theta + j\sin\theta), r(\cos\theta - j\sin\theta)$
  - Zeros: 1, -1
- Phase response
  - $\angle H(\omega) = \frac{\angle B(\omega)}{\angle A(\omega)} = \theta_1(\omega) + \theta_2(\omega) - \theta_3(\omega) - \theta_4(\omega)$



Practical Filters

- **One-pole one-zero filters**
  - Leaky integrator
  - DC-removal filters
  - Bass / treble shelving filter (EQ)

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 \cdot z^{-1}}{1 + a_1 \cdot z^{-1}}$$

- **Biquad filters**
  - Reson filter
  - Band-pass / notch filters
  - Equalizers

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}}$$

- **Any high-order filter** can be factored into **a cascade of one-pole one-zero filters or bi-quad filters**

$$H(z) = H_o(z) \prod H_{bi}(z)$$

Digital Audio Effects

- Filters, Equalizers
  - One-pole/one-zero filter and bi-quad filters
  - Small number of coefficients and small size of delay elements
- Delay-based Effects
  - Chorus, Flanger
  - FIR or IIR filters
  - Small number of coefficients and a mid-to-long size of delay lengths
- Spatial Effects
  - HRTF, Reverb
  - FIR or IIR filters
  - A large number of coefficients and long delay lengths