CTP 431 Music and Audio Computing

Discrete Fourier Transform

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Outlines

- Fourier Series
- Fourier Transform
- Discrete-Time Fourier Transform
- Discrete Fourier Transform
  - Fast Fourier Transform

Fourier Series

- Recall the "modes" in oscillation: the periodic signal x(t) with period T can be represented as

$$x(t) = \frac{1}{N}\sum_{k=1}^{\infty} r_k \sin(2\pi kt/T)$$

  – Correct?

- General form of a periodic signal x(t) with period T
  – Add phase and D.C. offset

$$x(t) = a_0 + \frac{1}{N}\sum_{k=1}^{\infty} r_k \sin(2\pi kt/T + \phi(k))$$

$$x(t) = a_0 + \frac{1}{N}\sum_{k=1}^{\infty} (a_k \cos(2\pi kt/T) + b_k \sin(2\pi kt/T))$$

Fourier Series

- How can you get the coefficients?
  - Use the orthogonality of sinusoids

$$\int_{-T/2}^{T/2} \cos(2\pi mt/T)\sin(2\pi nt/T)dt = 0$$

$$\int_{-T/2}^{T/2} \cos(2\pi mt/T)\cos(2\pi nt/T)dt = \begin{cases} T\ (m=n) \\ 0\ (m \neq n) \end{cases}$$

- Coefficients

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cos(2\pi kt/T)dt$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt$$

$$b_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\sin(2\pi kt/T)dt$$

Fourier Transform

- What if the signal is not periodic?
  - An aperiodic signal can be approximated by $T \rightarrow \infty$
  - Angular frequency $\omega_k = 2\pi k / T = 2\pi(kF) \rightarrow \omega = 2\pi f$

Discrete frequency Continuous frequency

- The general form is converted to

$$x(t) = \int_0^{\infty} (A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t))d\omega$$

- The coefficients are to

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} x(t)\cos(\omega t)dt \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$$

Fourier Transform

- Can we represent the transform in a simpler form?
  - Combine A(w) and B(w) into a single term
  - Amplitude and phase are explicit
  - Explain the properties of Fourier transform easily

Fourier Transform

- Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

  - Proof) Taylor's series
  - If $\theta = \pi,\ e^{j\pi} + 1 = 0$ ("the most beautiful equation in math")

- Properties

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Fourier Transform

- Plugging Euler's identify in Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(A^{'}(\omega)\cos(\omega t) + B^{'}(\omega)\sin(\omega t))d\omega$$

$$A^{'}(\omega) = \pi A(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt \qquad B^{'}(\omega) = \pi B(\omega) = \int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$$

- Fourier Transform

$$F(\omega) = A^{'}(\omega) - jB^{'}(\omega) = \int_{-\infty}^{\infty} x(t)(\cos(\omega t) - j\sin(\omega t))dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$

Fourier Transform

- Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(A'(\omega)\cos(\omega t) + B'(\omega)\sin(\omega t))d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(A'(\omega)(\frac{e^{j\omega t} + e^{-j\omega t}}{2}) + B'(\omega)(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}))d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}(A'(\omega) - jB'(\omega))e^{j\omega t} + \frac{1}{2}(A'(\omega) + jB'(\omega))e^{-j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\frac{1}{2}F(\omega)e^{j\omega t} + \frac{1}{2}\overline{F(\omega)e^{j\omega t}})d\omega = \text{Real}\{\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega\}$$

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega}$$

Discrete-Time Fourier Transform (DTFT)

- DTFT
  - Time is sampled

$$F(\omega) = \sum_{-\infty}^{\infty} x(n) e^{-j\omega n}$$

- Inverse DTFT
  - $F(\omega)$ is periodic in frequency domain

$$x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega t} d\omega$$

Discrete Fourier Transform

- Now, what if the discrete signal is finite in length (*N*) ?
  - This is the signal that we really handle

$$x(n) = [x_0, x_1, x_2, \cdots, x_{N-1}]$$

- <u>We assume that x(n) is periodic with period *N*</u>
  - Periodic in time → Sampling in frequency

$$\omega = 2\pi f \quad \longrightarrow \quad \omega_k = 2\pi k f_k = 2\pi k / N$$

Continuous frequency

Discrete frequency

Discrete Fourier Transform

- Discrete Fourier Transform

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = X_R(k) + jX_I(k)$$

  - Magnitude spectrum: $|X(k)| = A(k) = \sqrt{X_R^2(k) + X_I^2(k)}$
  - Phase spectrum: $\angle X(k) = \Theta(k) = \tan^{-1}(\frac{X_I(k)}{X_R(k)})$

- Inverse Discrete Fourier Transform

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

(Extra) Discrete Fourier Transform

- Can we represent x(n) with a finite set of sinusoids?
  - Finding $A(k), \phi(k)$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} A(k)\cos(2\pi kn/N + \phi(k))$$

- Orthogonality of Sinusoids
  - Inner product between two sinusoids

$$\sum_{n=0}^{N-1}\cos(2\pi pn/N)\cos(2\pi qn/N)) = \begin{cases} N/2 & \text{if } p = q \text{ or } p = N-q \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n=0}^{N-1}\cos(2\pi pn/N)\sin(2\pi qn/N)) = 0$$

$$\sum_{n=0}^{N-1}\sin(2\pi pn/N)\sin(2\pi qn/N)) = \begin{cases} 0 & \text{otherwise} \\ N/2 & \text{if } p = q \\ -N/2 & \text{if } p = N-q \end{cases}$$

(Extra) Discrete Fourier Transform

- Do the inner product with the signal and sinusoids

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1}(X_R(k)\cos(2\pi kn/N) - X_I(k)\sin(2\pi kn/N))$$

$$X_R(k) = A(k)\cos\Theta(k)$$

$$X_I(k) = A(k)\sin\Theta(k)$$

$$X_R(k) = \sum_{n=0}^{N-1} x(n)\cos(2\pi kn/N)$$

$$X_I(k) = -\sum_{n=0}^{N-1} x(n)\sin(2\pi kn/N)$$

We assume that $A(k) = A(N-k)$

- Using Euler's Identity

$$X(k) = X_R(k) + jX_I(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

(Extra) Discrete Fourier Transform

- Now the inverse discrete Fourier transform is derived as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} A(k)\cos(2\pi kn / N + \Theta(k))$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} A(k)(e^{j(2\pi kn/N+\Theta(k))} + e^{-j(2\pi kn/N+\Theta(k))}) / 2$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} (X(k)e^{j2\pi kn/N} + \overline{X(k)}e^{-j2\pi kn/N}) / 2$$

$$= \mathrm{Real}\{\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}$$

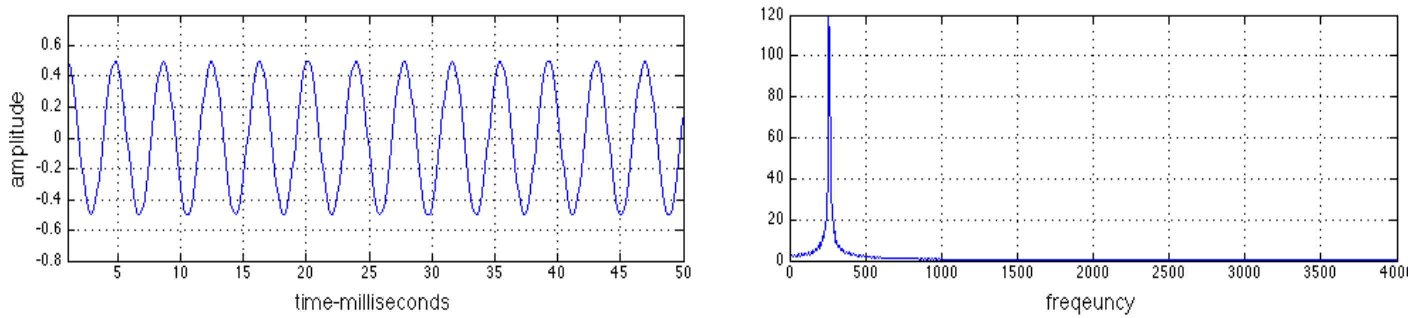
Fast Fourier Transform

- Matrix multiplication view of DFT

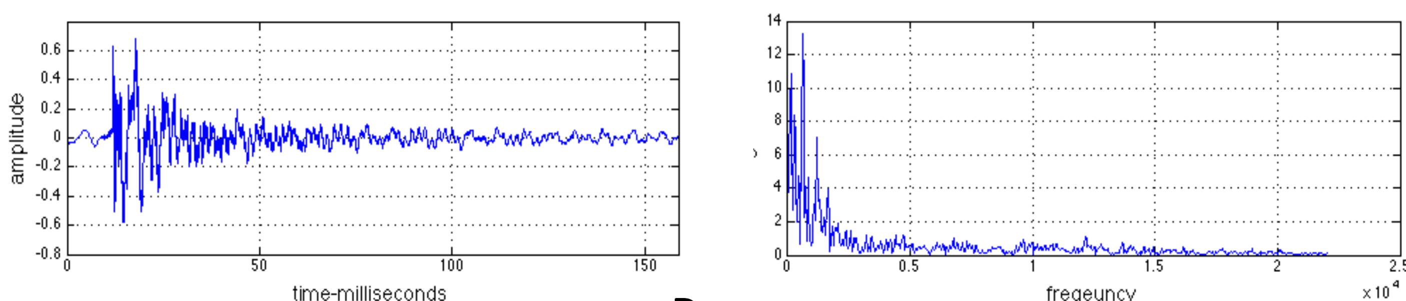
$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} s_0^*(0) & s_0^*(1) & \cdots & s_0^*(N-1) \\ s_1^*(0) & s_1^*(1) & \cdots & s_1^*(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ s_{N-1}^*(0) & s_{N-1}^*(1) & \cdots & s_{N-1}^*(N-1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

- In fact, we don't compute this directly. There is a more efficiently way, which is called "Fast Fourier Transform (FFT)"
  – Complexity reduction by FFT: $O(N^2) \rightarrow O(N\log_2 N)$
  – Divide and conquer

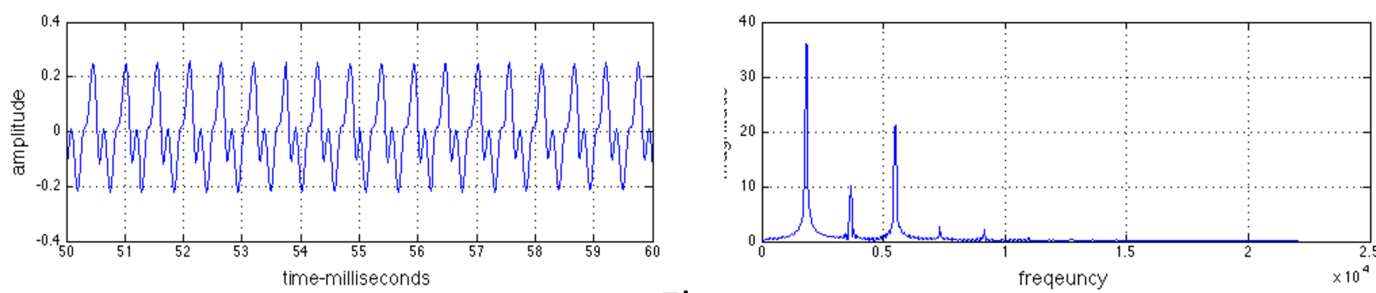
Examples of DFT



Sine waveform



Drum



Flute

Properties of DFT

- Linearity: $ax_1(n)+bx_2(n) \leftrightarrow aX_1(k)+bX_2(k)$

- Shift: $x(n-m) \leftrightarrow e^{-j2\pi mk/N}X(k)$

- Modulation (frequency shift): $e^{j2\pi mn/N}x(n) \leftrightarrow X(k-m)$

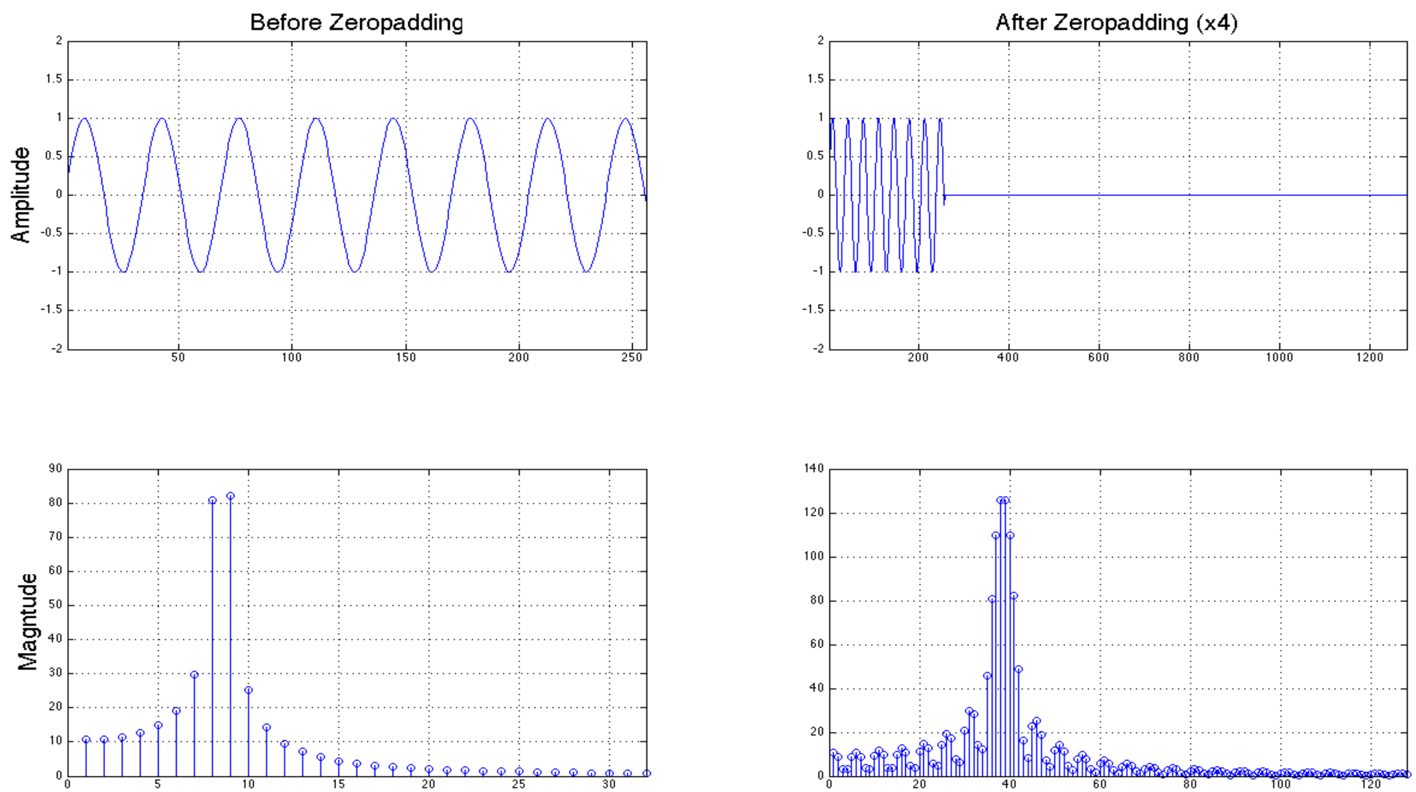
- Symmetry
  - If $x(n)$ is real, the magnitude is even-symmetry and the phase is odd-symmetry

- Convolution: $x_1(n)*x_2(n) \leftrightarrow X_1(k)X_2(k)$

  $x_1(n)x_2(n) \leftrightarrow X_1(k)*X_2(k)$ (Duality)

Zero-padding

- Adding zeros to a windowed frame in time domain
  - Corresponds to “ideal interpolation” in frequency domain
  - In practice, FFT size increases by the size of zero-padding



Demo: Fourier Series

- Web Audio Demo
  - http://codepen.io/anon/pen/jPGJMK (additive synthesis)