Variable Fractional Delay Filters in Bandlimited Oscillator Algorithms for Music Synthesis

(Invited Paper)

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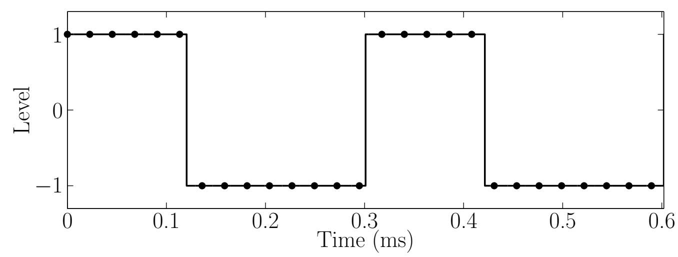
***Abstract*—Trivially sampled geometric waveforms such as the rectangular pulse wave used in subtractive sound synthesis suffer from aliasing caused by the discontinuities in the waveform or its derivative. Several algorithms for the reduction of aliasing distortion have been suggested, providing either complete removal or great suppression of aliasing. Some antialiasing oscillators utilize variable fractional delay filters as an essential part of the algorithm. In this paper, these oscillators are reviewed with an emphasis on motivating the use of the fractional delay filters.**

***Index Terms*—Fractional delay filters, bandlimited oscillators, music synthesis**

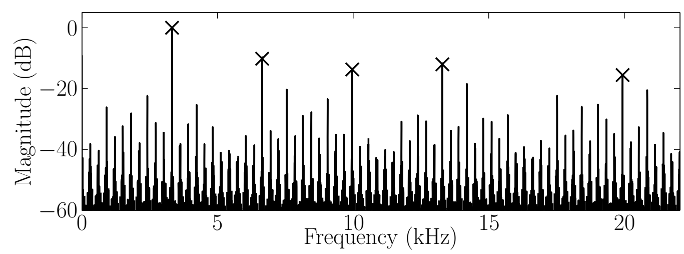
I. Introduction

Subtractive sound synthesis was a popular technique in music synthesizers in the 1960s and 1970s. In subtractive synthesis a spectrally rich source signal is shaped with a time-varying resonant filter to produce evolving and lively sounds. Today, digital emulation of these vintage sounds, often referred as "virtual analog synthesis," is an interesting topic of research, initiated in 1995 by the introduction of the Nord Lead synthesizer that emulated the analog synthesizers as a digital unit.

Traditionally, the subtractive synthesis technique uses periodic geometric waveforms such as sawtooth, rectangular pulse, and triangular pulse waves as the source signal [1], [2]. Unfortunately, as these classical waveforms have discontinuities in the waveform or waveform derivative, they have a non-bandlimited spectrum. Therefore, trivial digital synthesis of these waveforms via sampling results in an aliased signal in which the harmonics above the Nyquist limit fold back to the audio band [3]–[6]. Since the harmonics of these waveforms decay moderately, by about 6 dB per octave in case of a sawtooth and a rectangular pulse wave and by 12 dB per octave in case of a triangular pulse wave, the aliasing becomes clearly audible especially at high fundamental frequencies. The aliasing issue is illustrated in Fig. 1 where the waveform of a continuous-time rectangular pulse wave having fundamental frequency $f_0 = 3.322$ kHz (note G#7) and duty cycle of 40%



(a)



(b)

Fig. 1. (a) The continuous-time waveform of a rectangular pulse wave having fundamental frequency $f_0 = 3.322$ kHz (note G#7) and duty cycle of 40% and (b) the spectrum of its trivially sampled digital representation. The dots in (a) represent the sampled data. A sampling frequency $f_\mathrm{s} = 44.1$ kHz was used. The non-aliased spectral components are indicated with crosses in (b).

is plotted together with the spectrum of its trivially sampled digital representation for the sampling frequency $f_\mathrm{s} = 44.1$ kHz.

Numerous algorithms have been suggested to either remove or suppress the aliasing. Some of these algorithms produce an ideally bandlimited waveform by synthesizing a fixed number of harmonics [7]–[11]. Another group of algorithms allow some aliasing mainly at high frequencies where the human hearing is less sensitive by performing lowpass filtering prior to the sampling of the waveform. The lowpass filtering can be applied to the derivative of the waveform which is an impulse train [12] that is integrated after the filtering to obtain the bandlimited waveform. It has been also suggested that the integration can be performed beforehand in order to avoid numerical problems associated with the integration [13].

There are also algorithms that produce aliasing in the whole audio band but more suppressed than in the case of trivial sampling. The aliasing can be suppressed by sampling the trivial waveform at a higher sampling frequency [4], [6] but a very high oversampling factor should be used in order to obtain a proper alias reduction performance. Alternatively, the sawtooth waveform can be obtained by applying a tracking highpass and a fixed lowpass filter to a full-wave rectified sinusoid of half of the fundamental frequency [14] or by applying a differentiating filter to a piecewise polynomial waveform [15]–[18]. By modifying the basic algorithm of the latter approach the other classical waveforms can be obtained.

There are also other nonlinear techniques that can produce the classical waveforms [19]–[26] but it should be noted that these nonlinear approaches are generally not bandlimited. The aliasing can also be suppressed in the digital domain by applying a digital post-processing filter to the alias-corrupted waveform [27].

In this paper, the connection between the bandlimited interpolation via fractional delay filtering and bandlimited classical waveform synthesis is explained. Section II presents an antialiasing oscillator approach where the formulation of the algorithm resembles the formulation of the bandlimited interpolation. In Section III, the use of the fractional delay filters in the oscillator algorithm is investigated in more detail. Finally, Section IV concludes the paper.

II. Quasi bandlimited waveform synthesis

Of the bandlimited oscillator algorithms listed above, bandlimited interpolation and fractional delay filters are directly applicable to the wavetable synthesis approach [10] where the waveform is generated from a precomputed table that contains one cycle of bandlimited oscillation by stepping the table index appropriately to match the desired fundamental frequency. As the step size is in general not an integer, wavetable interpolation is needed to avoid deviation from the desired pitch. In addition, fractional delay filters are needed in the comb filters of the post-processing approach [27]. On the other hand, the phase distortion approaches that use time-varying first-order allpass filters [21]–[25] can be loosely interpreted as a special use of the first-order Thiran allpass fractional delay filter.

The Stilson and Smith approach [12] where the waveform derivative is bandlimited results in a formulation that resembles the formulation of the bandlimited interpolation. Consider, for instance, a continuous-time rectangular pulse wave having a duty cycle, i.e. the pulse width, P. It can be expressed as

$$r(t;P) = 2\sum_{k=-\infty}^{\infty}\left[u(t-kT_0) - u(t-(k+P)T_0)\right] - 1, \quad (1)$$

where $T_0 = 1/f_0$ is the oscillation period in seconds and $u(x)$ is the Heaviside unit step function,

$$u(x) = \begin{cases} 0, & \text{for } x < 0, \\ 0.5, & \text{for } x = 0, \text{ and} \\ 1, & \text{for } x > 0. \end{cases} \quad (2)$$

The derivative of the rectangular pulse wave is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} r(t;P) = 2\sum_{k=-\infty}^{\infty}\left[\delta(t-kT_0) - \delta(t-(k+P)T_0)\right], \quad (3)$$

where $\delta(x)$ is the Dirac delta (impulse) function that is zero when $x \neq 0$ and that satisfies the condition

$$\int_{-\infty}^{\infty}\delta(x)\mathrm{d}x = \int_{0-}^{0+}\delta(x)\mathrm{d}x = 1. \quad (4)$$

When the differentiated signal is bandlimited, each Dirac delta function is replaced with the impulse response of the bandlimiting lowpass filter [12], [28]. In the ideal case, the lowpass filter impulse response is given as

$$h_{\mathrm{id}}(t) = \mathrm{sinc}(f_{\mathrm{c}}t), \quad (5)$$

where f_{c} is the cutoff frequency of the filter and $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$.

Now, by integrating the bandlimited impulse train where the impulses of Eq. (3) are replaced with the bandlimited impulse of Eq. (5), the bandlimited rectangular pulse wave will be obtained. This approach is directly applicable also for sawtooth and triangular pulse waves. It should be noted that with the triangular pulse wave the waveform derivative is the rectangular pulse wave that has a duty cycle of 50% and an amplitude $8f_0A$ where A is the original amplitude. The Stilson and Smith approach, often called the bandlimited impulse train (BLIT) synthesis method, can therefore be depicted with a general block diagram given in Fig. 2. The phase counter steps the phase of the oscillator according to the desired fundamental and sampling frequencies. The discontinuity detector triggers the BLIT synthesizer whenever the waveform derivative contains a discontinuity. Finally, the bandlimited impulse train is integrated, or double integrated in the case of triangular pulse wave, to obtain the bandlimited waveform.

Three observations can be made from the derivation given above. First, as the fundamental frequency of the oscillation is arbitrary, waveform discontinuities are generally located between sampling instants and the location varies from discontinuity to another. This means that the peak of the sinc function needs to be shifted according to a *variable* fractional delay relative to the sampling instant following the location of the discontinuity. Second, since the sinc function is infinitely long, complete synthesis of the ideal bandlimited impulse train is impossible as it requires infinitely many sinc-function values to be summed at each sampling instant [12], [28]. Third, since the inline evaluation of the sinc function is impractical, it needs to be approximated. Pulses obtained with, e.g., modified FM synthesis [29] could be used but typically the sinc function is windowed and sampled from a table [30]. These approximations produce waveforms that are not completely bandlimited,

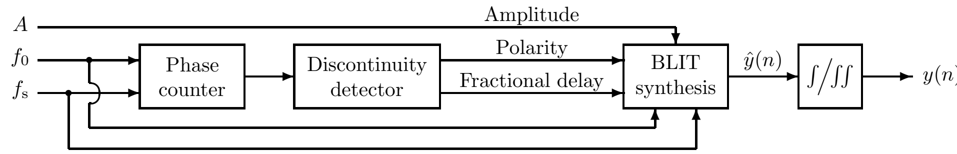


Fig. 2. Block diagram of an oscillator based on the BLIT approach. The bandlimited waveform $y(n)$ is obtained as the integral (double integral in case of triangular pulse wave) of the bandlimited impulse train $\hat{y}(n)$.

having some aliasing at high frequencies. However, since human hearing is less sensitive at high frequencies, the aliasing is not that clearly audible. Therefore, these algorithms are usually called quasi bandlimited oscillator algorithms [30].

III. Fractional delay filters in quasi bandlimited oscillator algorithms

As the basis function of ideal bandlimited interpolation is also the sinc function [31], [32], the approximation approaches used for bandlimited interpolation can be utilized also in the synthesis of a bandlimited impulse train. Practical and efficient approximation techniques for bandlimited interpolation are fractional delay filters [33], [34], and recently their use have been tested also in the synthesis of the bandlimited impulse train [35], [36].

A. Feedback delay loop oscillator

In [35] a low-order Thiran allpass filter was utilized in a feedback delay loop to generate an impulse train that has a desired constant period. The impulse train synthesis is triggered only once and the resulting impulse train is completely free from aliasing. The block diagram of a feedback delay loop BLIT synthesizer is depicted in Fig. 3. The delay line length is the integer part of the oscillation period $P_0 = f_{\rm s}/f_0$ and the fractional delay filter $H_{\rm fd}(z)$ is to be designed to implement the fractional part of the oscillation period $P_0 - \lfloor P_0 \rfloor$ at the fundamental frequency. A bipolar BLIT is obtained by using two feedback delay loops with the second loop, i.e., the loop that generates the negative pulses, triggered delayed with respect to the other. However, if the time difference between the positive and negative pulses is symmetric, i.e., the duty cycle is equal to 50%, the structure given in Fig. 3 can be used but with the loop delay halved and with the summation element replaced with a subtraction.

The waveform and the spectrum obtained with the feedback delay loop approach are presented in Fig. 4. The initial waveform (Fig. 4(a)) has a series of impulses with slight dispersion whereas the waveform after one second (Fig. 4(b)) completely becomes dispersed over time due to the frequency-dependent group delay of the allpass filter. This dispersion causes a small degree of inharmonic overtones as shown in Fig. 4(c) where the harmonic references are marked with crosses for comparison. However, an objective evaluation based on inharmonicity coefficient has showed the effect of inharmonicity to be nearly inaudible, although a slight time-varying phase shift was detected at high fundamental frequencies in an informal listening test [35].

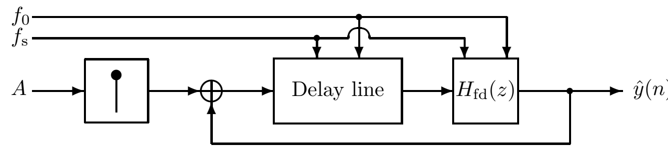


Fig. 3. Block diagram of the unipolar feedback delay loop BLIT algorithm [35]. The delay line length is the integer part of the oscillation period. The filter $H_{\rm fd}(z)$ is an allpass filter that produces the fractional delay. The block connected to the summation element is an impulse trigger.

When the fundamental frequency changes over time, for example, for vibrato and glissando, the delay produced by the feedback delay loop is required to change in real-time. However, with a single read pointer structure of Fig. 3 the changes in the delay line length and in the fractional delay filter coefficients can introduce discontinuities into the loop. This problem can be avoided by using two reading pointers corresponding to two different frequencies and cross-fading the two allpass filter outputs [37]. Van Duyne et al. suggested setting the range of the fractional delay between 0.618 and 1.618 to minimize the transient effect and modifying the cross-fader to ramp up after 5 warm-up samples which corresponds to the time that the transient dies out [38]. Fig. 5 shows the block diagram of the feedback delay loop algorithm that incorporates these extensions that enable time-varying pitch.

B. Fractional delay BLIT oscillators

In [36], bandlimited impulses were synthesized using low-order Lagrange and B-spline interpolators and Thiran allpass filters. These interpolators were also used in the fractional delay filter approaches to the bandlimited interpolation [33], [39]–[43]. Lagrange interpolation provides a maximally flat approximation of the sinc function around the zero frequency [33], [44]. B-splines are often used in image processing applications [42]. Thiran allpass filters have unity gain at all frequencies and they produce a maximally flat group delay response at DC [33].

The use of fractional delay filtering approaches is illustrated in Fig. 6 where the synthesis of the bandlimited impulse train, the resulting approximately bandlimited waveform, and the spectrum of the waveform are shown for the third-order Lagrange and B-spline interpolators. The continuous-time interpolation polynomials are shown in the impulse-train plot together with the fractional delay associated with each discontinuity. Note that the fractional delay varies from a discontinuity to another. It can also be seen that the aliasing is greatly reduced (compare Fig. 1(b) with Figs. 6(e) and 6(f))

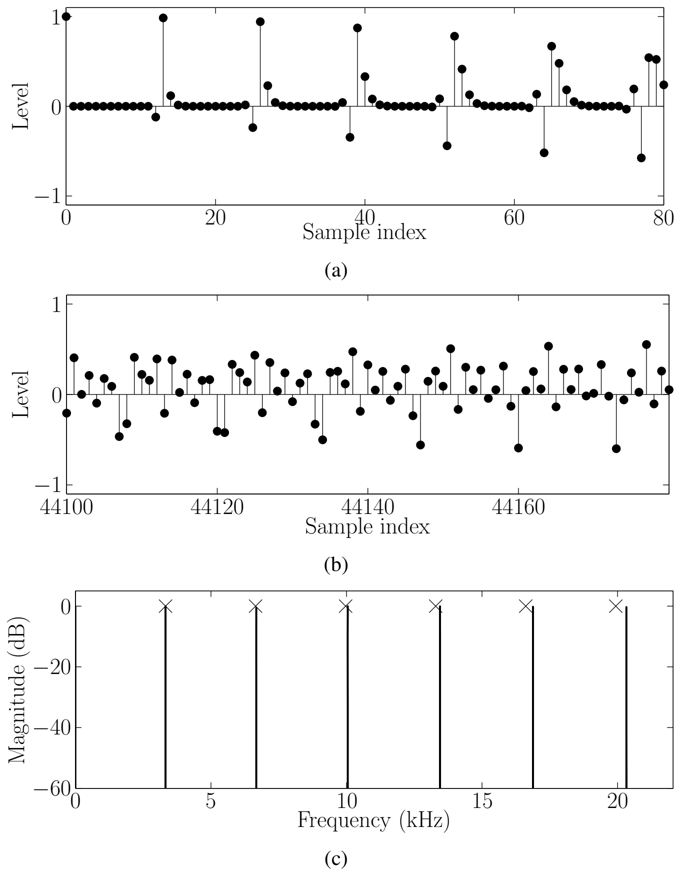


Fig. 4. The unipolar impulse train of the feedback delay loop BLIT approach after (a) the trigger and (b) one second of oscillation for $f_0 = 3.322$ kHz and $f_{\mathrm{s}} = 44.1$ kHz. The spectrum of the impulse train is shown in (c). It is clearly shown that the impulse train does not include any aliasing and that it is inharmonic (the desired harmonics are indicated with crosses in (c)). The inharmonicity makes the output of the algorithm to deviate from an ideal impulse train as illustrated in (b).

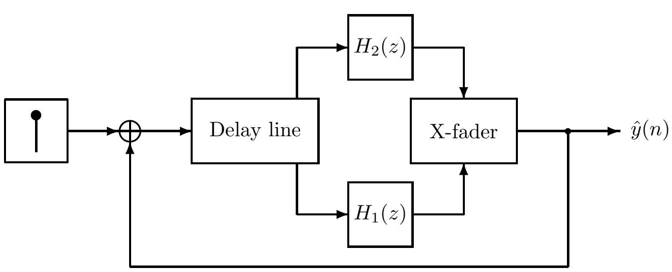


Fig. 5. Block diagram of the feedback delay loop BLIT algorithm that enables time-varying fundamental frequency [35]. The X-fader block performs the cross-fading of the allpass filter outputs.

especially at low frequencies.

It should also be noted that the peaks of the B-spline interpolation polynomials are less extreme than those of the Lagrange interpolator but since the B-spline interpolator is always non-negative, the integration of the impulse train results effectively in the same polarization change of the waveform. Moreover, the spectrum of the third-order B-spline given in Fig. 6(f) shows two interesting points. First, the third-order B-spline interpolator algorithm produces less aliasing compared to the Lagrange interpolator of same order (see Fig. 6(e)). Second, the amplitude drop of the higher harmonics is quite large with the B-spline interpolator whereas with the Lagrange interpolator the higher harmonics have almost the desired amplitudes. This happens because the B-spline interpolator has a rather steep roll-off to the stop-band with a trade-off of greater attenuation in the pass-band. By designing the fractional delay filter using another optimization criterion [33], a good alias reduction could be obtained while the pass-band attenuation would be reduced.

The low-order Thiran allpass fractional delay filters were noted to provide good alias reduction especially at low frequencies [36]. Up to the fundamental frequency the alias reduction obtained with the Nth-order allpass filter was observed to be close to that of the $(N+1)$th-order Lagrange interpolator for $N = 1$ and 2. However, the allpass filters produce considerable aliasing at higher frequencies and the aliasing level is clearly higher than with the Lagrange and B-spline interpolators. This is a result from the fact that the higher components are attenuated less with the allpass filters than with the polynomial interpolators. Still, the higher components are slightly attenuated also with the allpass fractional delay filters. Aside the higher aliasing level at high frequencies, the Thiran allpass filters have also another drawback. The computation of the filter coefficient requires a division (see [33]) every period which is impractical in hardware implementations. Approximations for the coefficient computation that avoid the division have been suggested in [35], [36].

As mentioned above, Brandt suggested in [13] that the numerical problems of the integration in the BLIT algorithm could be overcome by performing the integration before the sampling of the basis function. This means that the sampled sinc function is accumulated and used as an approximation of the bandlimited step function. Now, at every discontinuity the read of the bandlimited step function (BLEP) is triggered and output. When the table read exceeds the last element of the table, the synthesizer outputs ones. The algorithm can be further simplified by subtracting the unit step function from the table and applying the resulting BLEP residual as a correction function to the discontinuities of the trivial non-bandlimited waveform [30], [45] as illustrated in Fig. 7.

The BLEP approach can be applied to the fractional delay filtering BLITs as the BLEP function obtained with them is the integral of the filter coefficients with respect to the delay. This extension has been shown in [30] for the linear interpolator, i.e. the first-order Lagrange (and B-spline as they are exactly the same) interpolation filter. In [19] the extension was tested with a third-order spline and a truncated third-order Lagrange interpolators that modify only the two samples that precede and follow the discontinuity.

IV. Conclusions

Trivial sampling of classical geometric waveforms such as the sawtooth, rectangular pulse, and triangular pulse wave used in subtractive sound synthesis suffers from harsh aliasing caused by the discontinuities in the waveform or its derivative.

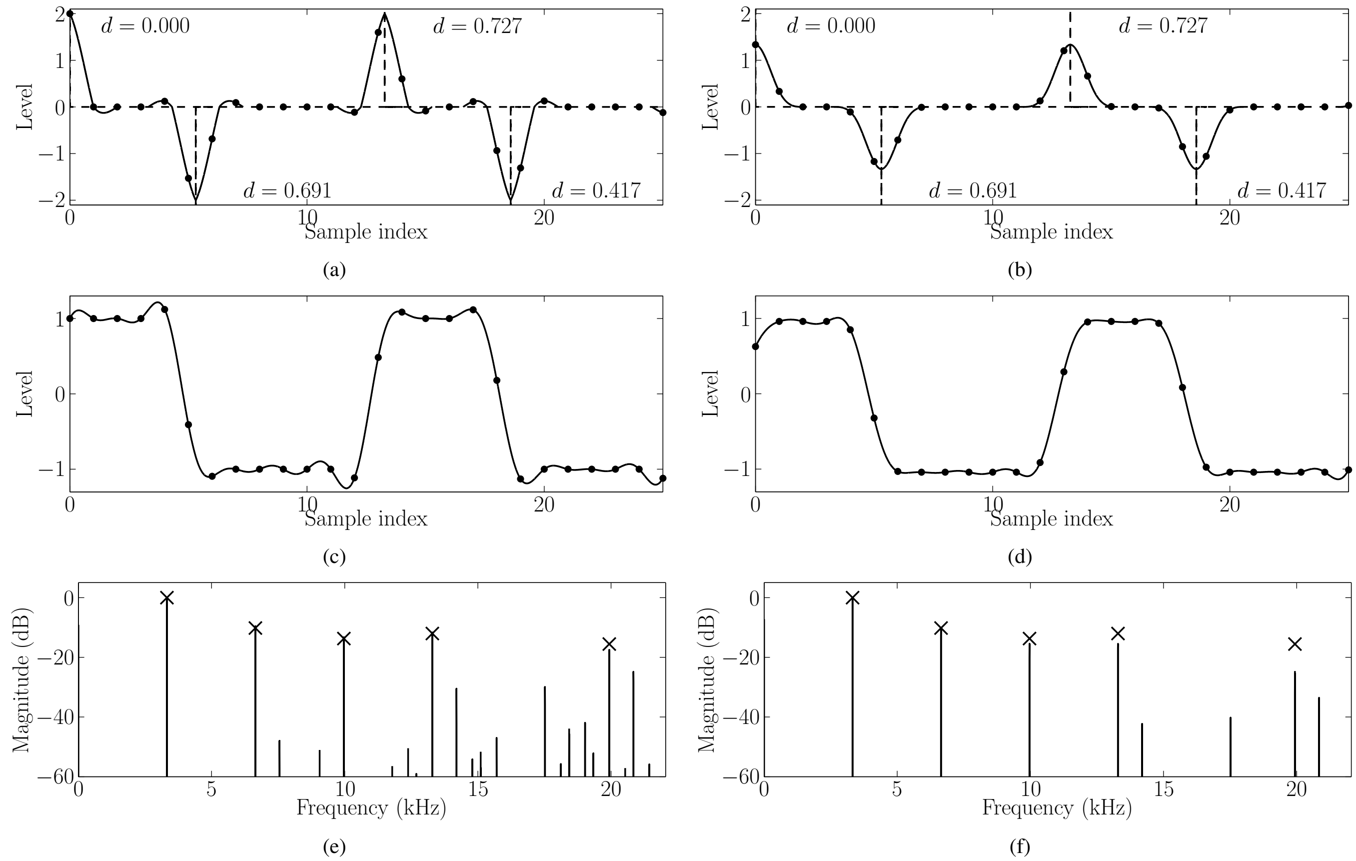


Fig. 6. The approximately bandlimited impulse train obtained with the third-order (a) Lagrange and (b) B-spline fractional delay filter for the rectangular pulse wave having duty cycle of 40%, $f_0 = 3.322$ kHz, and $f_s = 44.1$ kHz. The integrals of the bandlimited impulse trains, i.e., the approximately bandlimited rectangular pulse waves, are shown in (c) and (d) for the Lagrange and the B-spline approaches, respectively. The spectra of the waveforms are given in (e) for the Lagrange interpolator and in (f) for the B-spline interpolator. The dots in (a)–(d) represent the sampled data. In (a) and (b), the continuous-time impulse responses are shown with a solid line and the corresponding fractional delays of each discontinuity are given right to the impulses. The crosses in (e) and (f) illustrate the desired amplitudes of the non-aliased spectral components.

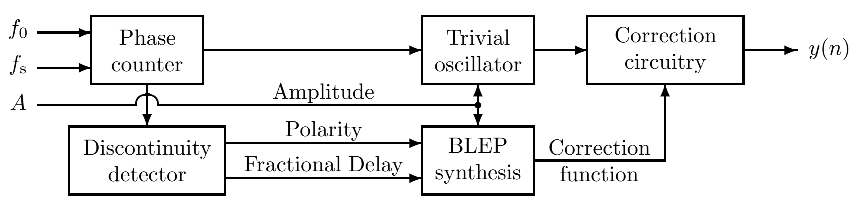


Fig. 7. Block diagram of an oscillator based on the BLEP approach. The bandlimited step function is applied as a correction function to the non-bandlimited trivial oscillator output.

Several techniques to remove or suppress the aliasing have been proposed and one of them formulates the bandlimited synthesis by applying a lowpass filter to the derivative of the waveform prior to sampling. As the derivative of the sawtooth and rectangular pulse waves and the second derivative of the triangular pulse wave is an impulse train, the impulses of the derivative are replaced with the impulse response of the lowpass filter. The integration of the resulting bandlimited impulse train (BLIT) produces a bandlimited waveform. In the ideal case, the BLIT formulation results in a bandlimited impulse that is the same as the basis function of the ideal bandlimited interpolation. Since ideal bandlimited interpolation is often approximated with fractional delay filters, it is natural to use them also to approximate the ideal bandlimited impulse of the BLIT approach. In this paper, this extension was reviewed with an emphasis on motivating the use of the fractional delay filters. Recently proposed feedback delay loop and fractional delay filter BLIT oscillators were presented in more detail and their alias reduction performance were exemplified.

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REFERENCES

[1] H. F. Olson and H. Belar, "Electronic music synthesis," *Journal of the Acoustical Society of America*, vol. 27, no. 3, pp. 595–612, May 1955.

[2] R. A. Moog, “Voltage-controlled electronic music modules,” *Journal of the Audio Engineering Society*, vol. 13, no. 3, pp. 200–206, July 1965.

[3] H. G. Alles, “Music synthesis using real time digital techniques,” *Proceedings of the IEEE*, vol. 68, no. 4, pp. 436–449, April 1980.

[4] H. Chamberlin, *Musical Applications of Microprocessors*, 2nd ed. Hasbrouck Heights, NJ: Hayden Book Company, 1985.

[5] F. R. Moore, *Elements of Computer Music*. Englewood Cliffs, NJ: Prentice Hall, 1990.

[6] M. Puckette, *The Theory and Technique of Electronic Music*. Hackensack, NJ: World Scientific Publishing Co., 2007, pp. 301–322.

[7] G. Winham and K. Steiglitz, “Input generators for digital sound synthesis,” *Journal of the Acoustical Society of America*, vol. 47, part 2, pp. 665–666, February 1970.

[8] J. A. Moorer, “The synthesis of complex audio spectra by means of discrete summation formulae,” *Journal of the Audio Engineering Society*, vol. 24, no. 9, pp. 717–727, November 1976.

[9] A. Chaudhary, “Bandlimited simulation of analog synthesizer modules by additive synthesis,” in *Proceedings of the Audio Engineering Society 105th Convention*, San Francisco, CA, September 1998, preprint number 4779.

[10] P. Burk, “Band limited oscillators using wave table synthesis,” in *Audio Anecdotes II — Tools, Tips, and Techniques for Digital Audio*, K. Greenebaum and R. Barzel, Eds. Wellesley, MA: A. K. Peters Ltd., 2004, pp. 37–53.

[11] G. Deslauriers and C. Leider, “A bandlimited oscillator by frequency-domain synthesis for virtual analog applications,” in *Proceedings of the Audio Engineering Society 127th Convention*, New York, NY, October 2009, preprint number 7923.

[12] T. S. Stilson and J. O. Smith, “Alias-free digital synthesis of classic analog waveforms,” in *Proceedings of the International Computer Music Conference*, Hong Kong, China, August 1996, pp. 332–335.

[13] E. Brandt, “Hard sync without aliasing,” in *Proceedings of the International Computer Music Conference*, Havana, Cuba, September 2001, pp. 365–368.

[14] J. Lane, D. Hoory, E. Martinez, and P. Wang, “Modeling analog synthesis with DSPs,” *Computer Music Journal*, vol. 21, no. 4, pp. 23–41, Winter 1997.

[15] V. Välimäki, “Discrete-time synthesis of the sawtooth waveform with reduced aliasing,” *IEEE Signal Processing Letters*, vol. 12, no. 3, pp. 214–217, March 2005.

[16] A. Huovilainen and V. Välimäki, “New approaches to digital subtractive synthesis,” in *Proceedings of the International Computer Music Conference*, Barcelona, Spain, September 2005, pp. 399–402.

[17] V. Välimäki and A. Huovilainen, “Oscillator and filter algorithms for virtual analog synthesis,” *Computer Music Journal*, vol. 30, no. 2, pp. 19–31, Summer 2006.

[18] V. Välimäki, J. Nam, J. O. Smith, and J. S. Abel, “Alias-suppressed oscillators based on differentiated polynomial waveforms,” *IEEE Transactions on Audio, Speech, and Language Processing*, vol. 18, no. 4, pp. 786–798, May 2010.

[19] J. Pekonen, “Computationally efficient music synthesis – methods and sound design,” Master of Science (Technology) thesis, TKK Helsinki University of Technology, Espoo, Finland, June 2007, available online at http://www.acoustics.hut.fi/publications/files/theses/pekonen_mst/ [Checked April 16, 2010].

[20] J. Kleimola, “Audio synthesis by bitwise logical modulation,” in *Proceedings of the 11th International Conference on Digital Audio Effects (DAFx-08)*, Espoo, Finland, September 2008, pp. 67–70.

[21] J. Timoney, V. Lazzarini, J. Pekonen, and V. Välimäki, “Spectrally rich phase distortion sound synthesis using an allpass filter,” in *Proceedings of the 2009 IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP'09)*, Taipei, Taiwan, April 2009, pp. 293–296.

[22] J. Timoney, V. Lazzarini, B. Carty, and J. Pekonen, “Phase and amplitude distortion methods for digital synthesis of classic analogue waveforms,” in *Proceedings of the Audio Engineering Society 126th Convention*, Munich, Germany, May 2009, preprint number 7792.

[23] J. Kleimola, J. Pekonen, H. Penttinen, V. Välimäki, and J. S. Abel, “Sound synthesis using an allpass filter chain with audio-rate coefficient modulation,” in *Proceedings of the 12th International Conference on Digital Audio Effects (DAFx-09)*, Como, Italy, September 2009, pp. 305–312.

[24] V. Lazzarini, J. Timoney, J. Pekonen, and V. Välimäki, “Adaptive phase distortion synthesis,” in *Proceedings of the 12th International Conference on Digital Audio Effects (DAFx-09)*, Como, Italy, September 2009, pp. 28–35.

[25] V. Lazzarini, J. Timoney, J. Kleimola, and V. Välimäki, “Five variations on a feedback theme,” in *Proceedings of the 12th International Conference on Digital Audio Effects (DAFx-09)*, Como, Italy, September 2009, pp. 139–145.

[26] V. Lazzarini and J. Timoney, “New perspectives on distortion synthesis for virtual analog oscillators,” *Computer Music Journal*, vol. 34, no. 1, pp. 28–40, Spring 2010.

[27] J. Pekonen and V. Välimäki, “Filter-based alias reduction for digital classical waveform synthesis,” in *Proceedings of the 2008 IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP'08)*, Las Vegas, NV, April 2008, pp. 133–136.

[28] T. S. Stilson, “Efficiently-variable non-oversampling algorithms in virtual-analog music synthesis – a root-locus perspective,” Ph.D. dissertation, Department of Electrical Engineering, Stanford University, Stanford, CA, June 2006, available online http://ccrma.stanford.edu/~stilti/papers/ [Checked April 16, 2010].

[29] J. Timoney, V. Lazzarini, and T. Lysaght, “A modified FM synthesis approach to bandlimited signal generation,” in *Proceedings of the 11th International Conference on Digital Audio Effects (DAFx-08)*, Espoo, Finland, September 2008, pp. 27–33.

[30] V. Välimäki and A. Huovilainen, “Antialiasing oscillators in subtractive synthesis,” *IEEE Signal Processing Magazine*, vol. 24, no. 2, pp. 116–125, March 2007.

[31] J. O. Smith and P. Gossett, “A flexible sampling-rate conversion method,” in *Proceedings of the 1984 IEEE International Conference on Audio, Speech, and Signal Processing (ICASSP'84)*, vol. 9, San Diego, CA, March 1984, pp. 112–115.

[32] J. O. Smith, *Digital Audio Resampling Home Page*. http://ccrma.stanford.edu/~jos/resample/resample.html [Checked April 16, 2010], 2002.

[33] T. I. Laakso, V. Välimäki, M. Karjalainen, and U. K. Laine, “Splitting the unit delay – tools for fractional delay filter design,” *IEEE Signal Processing Magazine*, vol. 13, no. 1, pp. 30–60, January 1996.

[34] V. Välimäki and T. I. Laakso, “Principles of fractional delay filters,” in *Proceedings of the 2000 IEEE International Conference on Audio, Speech, and Signal Processing*, vol. 6, Istanbul, Turkey, June 2000, pp. 3870–3873.

[35] J. Nam, V. Välimäki, J. S. Abel, and J. O. Smith, “Alias-free oscillators using feedback delay loops,” in *Proceedings of the 12th International Conference on Digital Audio Effects (DAFx-09)*, Como, Italy, September 2009, pp. 347–352.

[36] ——, “Efficient antialiasing oscillator algorithms using low-order fractional delay filters,” *IEEE Transactions on Audio, Speech, and Language Processing*, vol. 18, no. 4, pp. 773–785, May 2010.

[37] D. A. Jaffe and J. O. Smith, “Performance expression in commuted waveguide synthesis of bowed strings,” in *Proceedings of the International Computer Music Conference*, Banff, Canada, September 1995, pp. 343–346.

[38] S. A. Van Duyne, D. A. Jaffe, G. P. Scandalis, and T. S. Stilson, “A lossless, click-free, pitchbend-able, delay line loop interpolation scheme,” in *Proceedings of the International Computer Music Conference*, Thessaloniki, Greece, September 1997, pp. 252–255.

[39] R. W. Schafer and L. R. Rabiner, “A digital signal processing approach to interpolation,” *Proceedings of the IEEE*, vol. 61, no. 6, pp. 692–702, June 1973.

[40] O. D. Grace, “Polynomial interpolation of bandlimited signals,” *Journal of the Acoustical Society of America*, vol. 54, no. 3, pp. 807–808, September 1973.

[41] H. W. Strube, “Sampled-data representation of a nonuniform lossless tube of continuously variable length,” *Journal of the Acoustical Society of America*, vol. 57, no. 1, pp. 256–257, January 1975.

[42] M. Unser, “Splines: A perfect fit for signal and image processing,” *IEEE Signal Processing Magazine*, vol. 16, no. 6, pp. 22–38, November 1999.

[43] A. Franck, “Efficient algorithms and structures for fractional delay filtering based on Lagrange interpolation,” *Journal of the Audio Engineering Society*, vol. 56, no. 12, pp. 1036–1056, December 2008.

[44] E. Hermanowicz, “Explicit formulas for weighting coefficients of maximally flat tunable FIR delayers,” *Electronics Letters*, vol. 28, no. 20, pp. 1936–1937, September 1992.

[45] A. B. Leary and C. T. Bright, “Bandlimited digital synthesis of analog waveforms,” U.S. Patent 7,589,272, September 2009.